Mathematical Methods 2024

Meet The Assessors Presentation: Review of 2023 exam

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MAV Meet The Assessors 2024 MM

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- The MAV has made the 2023 MAV Solutions to 2023 VCAA Mathematical Methods exams resource available as downloadable files.
- For 2024 VCAA will provide suggested solutions and also workshops for MTA
- The files can be downloaded easily from the Thinkific platform

Message from MAV

- *“A copy of the presenters PowerPoint will be made available to all workshop session attendees. At the conclusion of the workshop series (towards the end of term 1) the PowerPoints will be uploaded to the MAV Thinkific platform. All participants will then receive an email (similar to that for Exam Solutions resources) from MAV (Danijela Draskovic) via Thinkific notifying them the PowerPoint resource has been uploaded to Thinkific and is available to download.”*

Scaling 2023

- Mean 34.3 (33.9 in 2022) and SD 8.5 (8.4 in 2022)
- 20 (21)
- 25 (28)
- 30 (35)
- 35 (41) up 1 from previous year
- 40 (46) up 1 from previous year
- 45 (49)
- 50 (51)

Mathematics:									
Foundation Mathematics	20.6	7.0	16	19	24	28	33	40	50
General Mathematics	27.7	7.2	18	22	27	33	38	44	50
Mathematical Methods	34.3	8.5	21	28	35	41	46	49	51
Specialist Mathematics	41.6	8.1	29	37	43	48	52	54	55

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- Scanned images are used with marking, so encourage students to write clearly and in dark pencil or pen.
- Pencil work– check that faint pencil is legible when scanned, especially with graph work.
- Finer details such as labelling graphs with intercepts– ensure that handwriting is legible, 3s can look like 5s and pi can look like 4
- Try to present the solution in the spaces provided or at least provide <u>clear</u> indications where the solution may be found.

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- Check the question is being answered correctly.
  - If a question asks for a rule, make sure the rule is given not just c=.
  - If a particular method is required it should be clear that the method has been used.
  - Always reread a question before moving on to make sure all requirements of a question have been met. Find max area, not just value that gives max. If a derivative value at a point is requested then further work beyond finding the derivative function is required
- When writing an interval check the appropriate brackets are used.

[-2,2] is not the same as (-2, 2) or [2,-2]. Make sure brackets are clearly one or the other. $[-1, \infty)$ is not the same as $(\infty, -1]$. Intervals without brackets are no longer intervals!

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- Ensure the correct labels of functions are used. If a question asks for $f'(x)$ be sure to answer as $f'(x)$ and not $y=$
- Show adequate working for questions worth more than one mark.
- Use the variables that are given within the question and if variables are introduced then define them (practise with SAC questions). Don't mix variables, $A'(k)$ should not have x
- Watch vinculums– horizontal not diagonal, extension

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- The proper representation of a logarithm has a base as a subscript, for example $\log_{10}(100)$. An integral term must be accompanied by a 'dx' as in $\int f(x)\,dx$. Intersection and union symbols.
- Don't assume steps in *Show That* questions. Make sure each step is clear, logical and explicit. Work one side only and show it leads to other.
- Use brackets correctly; good practice to put brackets around argument.
- Put units in the final answer, unless units are mentioned in question stem.
- Check that the final answer makes sense. p values and sin(x) within [-1,1]
- It is expected that students will have a way of remembering the exact values for sin, cos and tan for angles between 0 and pi/2.

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- Take time when drawing graphs – use grid lines, check if coordinates are required, one sharp, smooth line/curve…
- Understanding maths language: show, simplify, verify, coordinates
- Use of notation
  - $-1 \leq x < 4$ rather than $-1 \geq x > 4$
  - $(-\infty, 1] \cup (2,5]$ rather than $(-\infty, 1] \cap (2,5]$
  - $R\backslash\{0\}$ not $R/\{0\}$
- Arithmetic and calculations
- Formula sheet– derivatives, integrals, trapezium rule, Newtons method

Question 1a.

Marks	0	1	2	Average
%	8	50	42	1.3

a. Let $y = \dfrac{x^2 - x}{e^x}$.

Find and simplify $\dfrac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{e^x \cdot (2x-1) - e^x(x^2-x)}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{-x^2+3x-1}{e^x} \text{ or } \frac{-(x^2-3x+1)}{e^x} \text{ or } (-x^2+3x-1)e^{-x}.$$

Need to use quotient rule or product rule
Need to simplify– know what it means to simplify
Use of brackets to deal with terms and signs
Vinculum
Notation

Question 1b.

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Marks	0	1	2	Average
%	9	26	65	1.6

b. Let $f(x) = \sin(x)e^{2x}$.

Find $f'\left(\frac{\pi}{4}\right)$.

$$f'(x) = 2\sin(x)\,e^{2x} + \cos(x)\,e^{2x}$$

$$f'\left(\frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{4}\right)e^{\frac{\pi}{2}} + \cos\left(\frac{\pi}{4}\right)e^{\frac{\pi}{2}}$$

$$f'\left(\frac{\pi}{4}\right) = \sqrt{2}\,e^{\frac{\pi}{2}} + \frac{\sqrt{2}\,e^{\frac{\pi}{2}}}{2} = \frac{3\sqrt{2}e^{\frac{\pi}{2}}}{2} \;or\; \frac{3e^{\frac{\pi}{2}}}{\sqrt{2}}$$

Use of product rule to differentiate
Need to then evaluate
Important to know exact values of basic angles
Practise dealing with surd terms
No requirement to simplify, but if do then it needs to be correct

Question 2.

Marks	0	1	2	3	Average
%	29	2	13	56	2.0

Question 2 (3 marks)

Solve $e^{2x} - 12 = 4e^x$ for $x \in R$.

$$e^{2x} - 4e^x - 12 = 0$$

$$m^2 - 4m - 12 = 0 \text{ where } m = e^x$$

$$(m-6)(m+2) = 0$$

$$m = 6 \text{ or } m = -2$$

$$e^x = 6 \text{ or } e^x = -2$$

$$\therefore\ x = \log_e 6$$

Set up a quadratic and solve.
NFL most efficient but could use quadratic formula to solve.
Remember that $\log_e(a)$ only exists for $a > 0$.

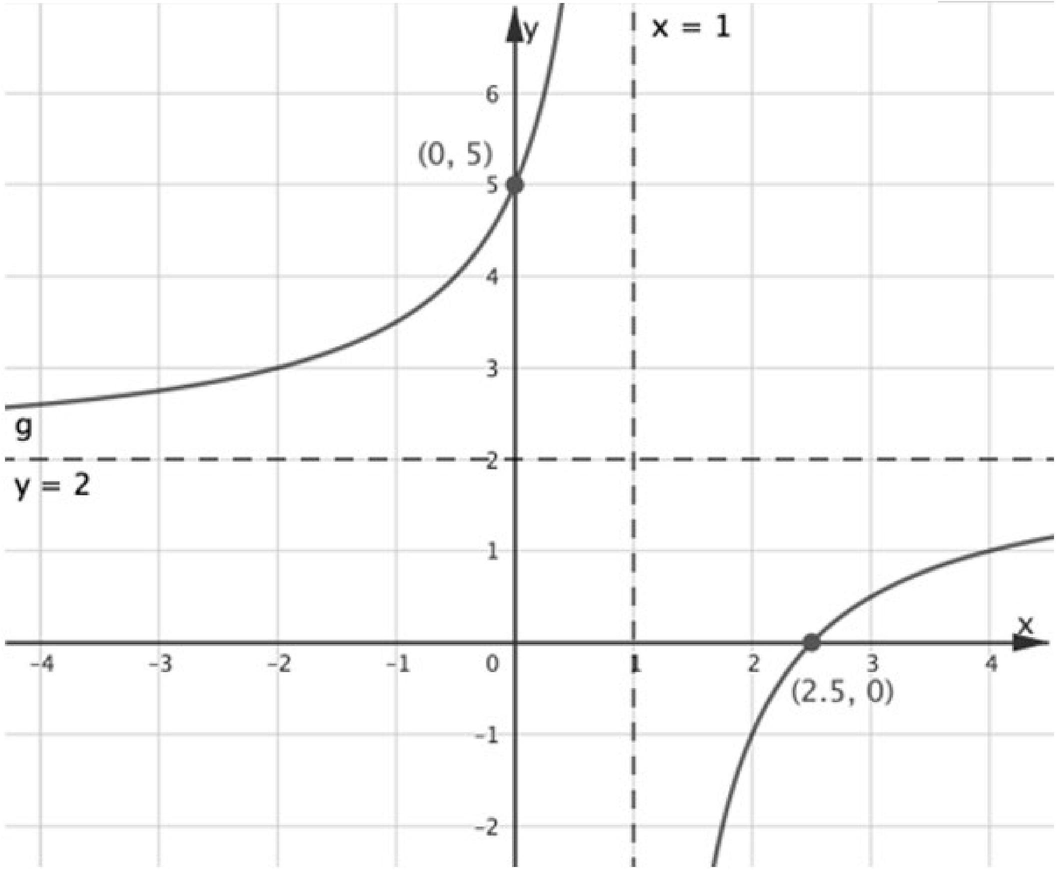
Question 3a.

Marks	0	1	2	3	Average
%	7	7	19	66	2.4

Sketch the graph of $f(x) = 2 - \dfrac{3}{x-1}$ on the axes below, labelling all asymptotes with their equations and axial intercepts with their coordinates.

3 marks

Graph work generally well done
Pay attention to grid lines, curvature, asymptotic behaviour and labels
Most common errors were labelling the y-intercept as (5,0), the x-intercept as $\left(\frac{3}{2}, 0\right)$, the vertical asymptote as y=1 and horizontal asymptote as x=2.
Watch for faint asymptotes, dotted/dashed lines, symmetry



Question 3b.

Marks	0	1	Average
%	62	38	0.4

b. Find the values of x for which $f(x) \leq 1$.

Use the graph from part 3a to assist
Watch brackets and endpoints
Most common incorrect answer was $(-\infty, 4]$

$1 < x \leq 4 \ or \ (1, 4]$

Question 4.

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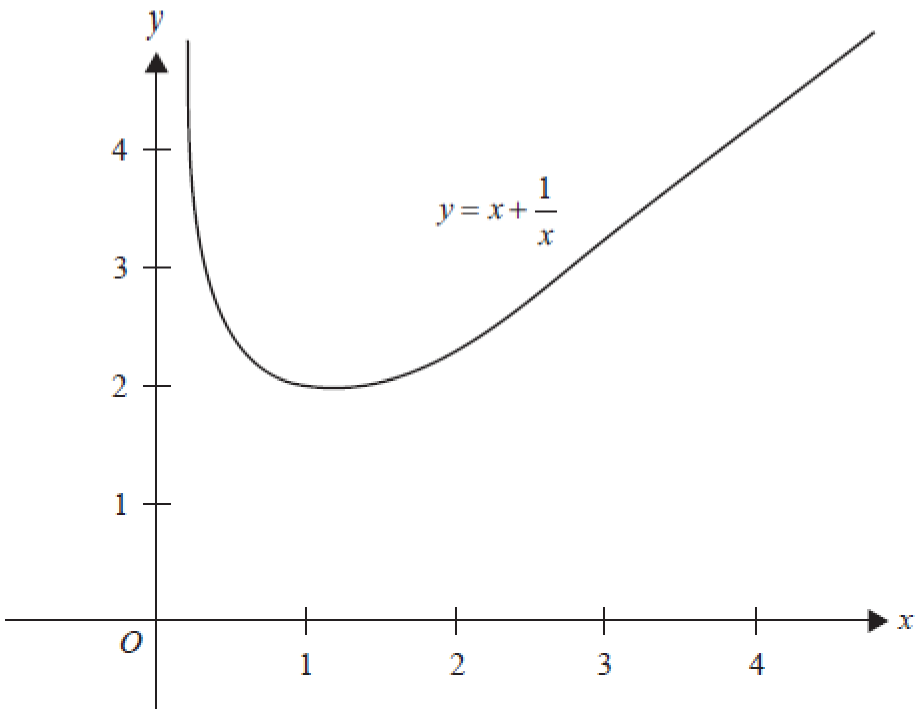
The graph of $y = x + \dfrac{1}{x}$ is shown over part of its domain.



Use two trapeziums of equal width to approximate the area between the curve, the x-axis and the lines $x = 1$ and $x = 3$.

Question 4.

Marks	0	1	2	Average
%	35	20	45	1.1

From the formula sheet:

trapezium rule approximation	$Area \approx \frac{x_n - x_0}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-2}) + 2f(x_{n-1}) + f(x_n)\right]$

$$f(1) = 1 + 1 = 2$$

$$f(2) = 2 + \frac{1}{2} = \frac{5}{2}$$

$$f(3) = 3 + \frac{1}{3} = \frac{10}{3}$$

Area = $\frac{3-1}{2\times 2}\left[2 + 2 \times \frac{5}{2} + \frac{10}{3}\right]$

$$= \frac{1}{2}\left[\frac{6}{3} + \frac{15}{3} + \frac{10}{3}\right]$$

$$= \frac{31}{6} = \frac{62}{12} = 5\frac{1}{6}$$

Must use an approximate method– area of trapezium not integration

See formula sheet and know how to use

Errors with evaluating f(2) and f(3) and then manipulation errors in dealing with the fractions

Clear setting out

Question 5a.

Marks	0	1	Average
%	34	66	0.7

a. Evaluate $\int_0^{\frac{\pi}{3}} \sin(x)dx$. 1 mark

Need to evaluate a definite integral
Antidifferentiate correctly and then evaluate
Errors with the antiderivative and with the evaluation

$$\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} \text{ instead of } \frac{1}{2}.$$

$$\frac{1}{2}$$

$$\begin{aligned}\int_0^{\frac{\pi}{3}} \sin(x)\, dx &= [-\cos(x)]_0^{\frac{\pi}{3}} \\ &= -\cos\left(\frac{\pi}{3}\right) - (-\cos(0)) \\ &= -\frac{1}{2} + 1 \\ &= \frac{1}{2}\end{aligned}$$

Question 5b.

Marks	0	1	2	3	Average
%	22	17	27	35	1.7

b. Hence, or otherwise, find all values of k such that $\int_0^{\frac{\pi}{3}} \sin(x)dx = \int_k^{\frac{\pi}{2}} \cos(x)dx$, where $-3\pi < k < 2\pi$. 3 marks

$$\int_k^{\frac{\pi}{2}} \cos(x)\,dx = [\sin(x)]_k^{\frac{\pi}{2}}$$
$$= \sin\left(\frac{\pi}{2}\right) - \sin(k)$$
$$= 1 - \sin(k)$$

Hence, although not essential
Allowed students opportunity to spot error from part a since $\sin(x)$ lies within $[-1, 1]$, $\sin(k)$ can never be $\frac{3}{2}$

Using part a.,

$$\frac{1}{2} = 1 - \sin(k)$$
$$\sin(k) = \frac{1}{2}$$
$$k = \frac{-11\pi}{6}, \frac{-7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$$

Question 6a.

Marks	0	1	Average
%	48	52	0.5

Let $\hat{P}$ be the random variable that represents the sample proportion of households in a given suburb that have solar panels installed.

From a sample of randomly selected households in a given suburb, an approximate 95% confidence interval for the proportion p of households having solar panels installed was determined to be $(0.04, 0.16)$.

a. Find the value of $\hat{p}$ that was used to obtain this approximate 95% confidence interval. 1 mark

$$(0.04 + 0.16) \div 2 = 0.1$$

$$\hat{p} = 0.1$$

Errors included $\hat{p}$ values greater than one; students are reminded that $0 \leq \hat{p} \leq 1$.

Question 6b

Marks	0	1	2	Average
%	51	23	26	0.7

Use $z = 2$ to approximate the 95% confidence interval.

b. Find the size of the sample from which this 95% confidence interval was obtained. 2 marks

b. Using either the upper end or lower end of the confidence interval to get

$$0.04 = 0.1 - 2 \times \sqrt{\frac{0.1 \times 0.9}{n}} \quad \text{or} \quad 0.16 = 0.1 + 2 \times \sqrt{\frac{0.1 \times 0.9}{n}}$$

$$0.06 = 2 \times \sqrt{\frac{0.1 \times 0.9}{n}}$$

$$0.03 = \sqrt{\frac{0.1 \times 0.9}{n}}$$

$$(0.03)^2 = \frac{0.1 \times 0.9}{n} = \frac{9}{100n} \quad \text{so} \quad \frac{9}{10000} = \frac{9}{100n}$$

$$n = 100$$

Use either the upper or lower bound of the confidence interval to form an equation
arithmetic

Question 6c.

Marks	0	1	Average
%	77	23	0.2

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A larger sample of households is selected, with a sample size four times the original sample. The sample proportion of households having solar panels installed is found to be the same.

By what factor will the increased sample size affect the width of the confidence interval? 1 mark

Width of the confidence interval is proportional to $\frac{1}{\sqrt{n}}$. If n is increased by a factor of 4, then width of confidence interval is halved.

A factor of $\frac{1}{4}$ was a common incorrect answer.

Question 7a.

Marks	0	1	Average
%	10	90	0.9

Question 7 (7 marks)

Consider $f: (-\infty, 1] \rightarrow R, f(x) = x^2 - 2x$. Part of the graph of $y = f(x)$ is shown below.

a. State the range of f.

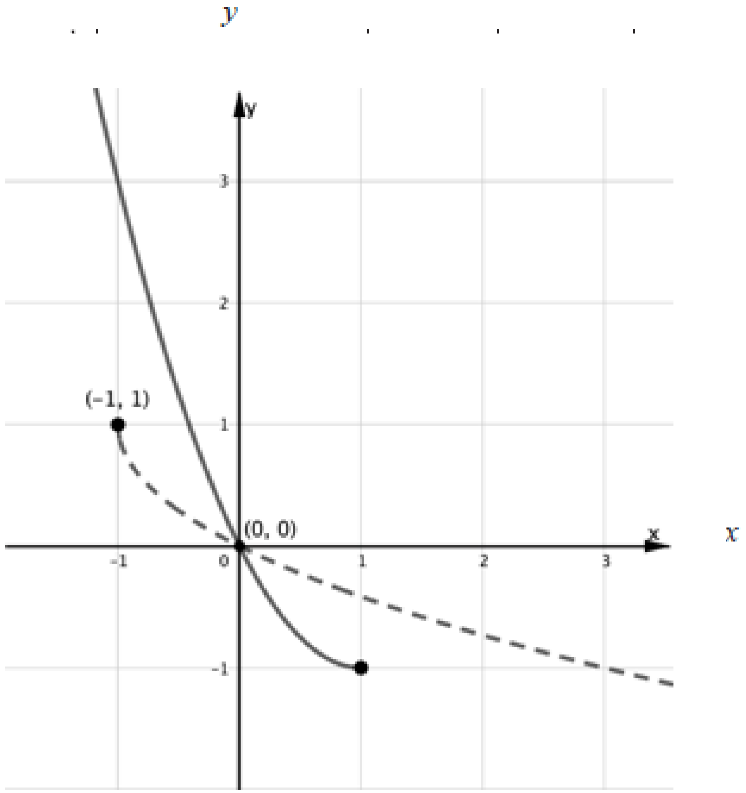
$[-1, \infty)$

The most common errors involved stating the range with incorrect brackets or swapping the ends of interval as in (∞,-1] .
Mathematical notation is a precise language.
Think about the order of terms- negative to positive

Question 7b.

Marks	0	1	2	Average
%	33	20	47	1.1

b. Sketch the graph of the inverse function $y = f^{-1}(x)$ on the axes above. Label any endpoints and axial intercepts with their coordinates. 2 marks



Graph of the inverse function is a reflection of the original in the line $y = x$

grid should serve as a guide to ensure the graph is correctly presented..

Question 7c.

Marks	0	1	2	Average
%	26	54	21	1.0

Determine the equation and the domain for the inverse function f^{-1}.

$$f^{-1}(x) = 1 - \sqrt{x+1}$$

Domain $[-1, \infty)$

Need to interchange x and y -- some difficulties in dealing with $x = y^2 - 2y$.

The domain of the inverse function being the range of the original was well recognised, independent of work in obtaining the equation of the function.

Most common error was writing the function as $f^{-1}(x) = 1 + \sqrt{x+1}$, the positive arm of the inverse.

Suggest use graph drawn in part 7b. to assist.

Use correct notation to denote the inverse function $f^{-1}(x)$.

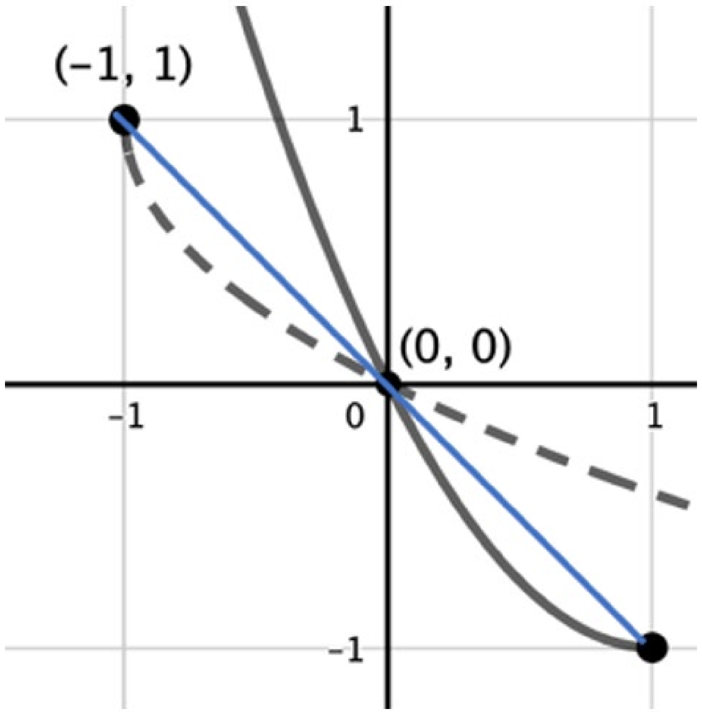
Question 7d.

Marks	0	1	2	Average
%	71	10	19	0.5

d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$.

There are many ways to approach this question. Using symmetry assists with the calculation.

Three suggested approaches are:



Question 7d.

Marks	0	1	2	Average
%	71	10	19	0.5

d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$.

Method 1

Using symmetry and either $y = -x$ and $f(x)$ or $y = -x$ and $f^{-1}(x)$

$$A = 2\int_0^1 \left(-x - (x^2 - 2x)\right)dx$$

$$A = 2\int_0^1 (x - x^2)\,dx$$

$$A = 2\left[\frac{1}{2}x^2 - \frac{x^3}{3}\right]_0^1$$

$$A = 2\left(\frac{1}{2} - \frac{1}{3} - (0)\right)$$

$$A = \frac{1}{3}$$

Question 7d.

Marks	0	1	2	Average
%	71	10	19	0.5

d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$.

Method 2

Finding the area of the two separate regions

$$A = \int_{-1}^{0} \left(-x - \left(1 - \sqrt{x+1}\right)\right) dx + \int_{0}^{1} \left(-x - (x^2 - 2x)\right) dx$$

$$A = \left[-\frac{x^2}{2} - x + \frac{2}{3}(x+1)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\frac{1}{2}x^2 - \frac{x^3}{3}\right]_{0}^{1}$$

$$A = \left(0 - 0 + \frac{2}{3} - \left(-\frac{1}{2} + 1 + 0\right)\right) + \left(\frac{1}{2} - \frac{1}{3} - 0\right)$$

$$A = \frac{1}{3}$$

Question 7d.

Marks	0	1	2	Average
%	71	10	19	0.5

d. Calculate the area of the regions enclosed by the curves of f, f^{-1} and $y = -x$.

Method 3

Using a triangle and the function

$$A = 2\left(-\frac{1}{2} - \int_0^1 (x^2 - 2x)dx\right)$$

$$A = 2\left(-\frac{1}{2} - \left[\frac{1}{3}x^3 - x^2\right]_0^1\right)$$

$$A = 2\left(-\frac{1}{2} - \left[\frac{1}{3} - 1\right]\right)$$

$$A = \frac{1}{3}$$

Using symmetry was a good option
Lack of brackets caused problems with signs
Some calculations did not involve y=-x

Question 8a.

Marks	0	1	Average
%	57	43	0.4

Suppose that the queuing time, T (in minutes), at a customer service desk has a probability density function given by

$$f(t) = \begin{cases} kt(16 - t^2) & 0 \le t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

for some $k \in R$.

a. Show that $k = \dfrac{1}{64}$.

$$\int_0^4 kt(16 - t^2)dt = 1$$

$$k\left[8t^2 - \frac{t^4}{4}\right]_0^4 = 1$$

$$k\left(8 \times 16 - \frac{16 \times 16}{4}\right) = 1$$

$$64k = 1$$

$$\therefore k = \frac{1}{64}$$

The total probability of the distribution needs to equal one.

Note: This is a 'show that' question so each step must be a clear and logical progression from the previous one.

Question 8b.

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Marks	0	1	2	Average
%	50	24	26	0.8

b. Find $\mathrm{E}(T)$.

$$E(T) = \frac{1}{64}\int_0^4 (16t^2 - t^4)dt$$

$$= \frac{1}{64}\left[\frac{16t^3}{3} - \frac{t^5}{5}\right]_0^4$$

$$= \frac{1}{64}\left(\frac{1024}{3} - \frac{1024}{5} - 0\right)$$

$$= \frac{1}{64} \times \frac{2048}{15}$$

$$= \frac{64}{30} = \frac{32}{15} = 2\frac{2}{15}$$

Take care with mathematical nomenclature– mixture of t and xs.

Arithmetic, work ways to deal with size of numbers

Question 8c.

Marks	0	1	2	3	Average
%	63	13	13	11	0.7

c. What is the probability that a person has to queue for more than two minutes, given that they have already queued for one minute?

$$\Pr(2 < T < 4|T > 1) = \frac{Pr(2<T<4)}{\Pr(T>1)}$$

$$= \frac{\frac{1}{64}\int_2^4 (16t-t^3)dt}{\frac{1}{64}\int_1^4 (16t-t^3)dt} = \frac{\int_2^4 (16t-t^3)dt}{\int_1^4 (16t-t^3)dt}$$

$$= \frac{\left[8t^2-\frac{t^4}{4}\right]_2^4}{\left[8t^2-\frac{t^4}{4}\right]_1^4}$$

$$= \frac{(64-(32-4))}{\left(64-\left(8-\frac{1}{4}\right)\right)}$$

$$= \frac{36}{\frac{225}{4}}$$

$$= \frac{144}{225} = \frac{16}{25} = 0.64$$

Recognise as conditional

Queue for one minute meant some wrote $\Pr(T > 2 \mid T = 1)$,

Errors with terminals

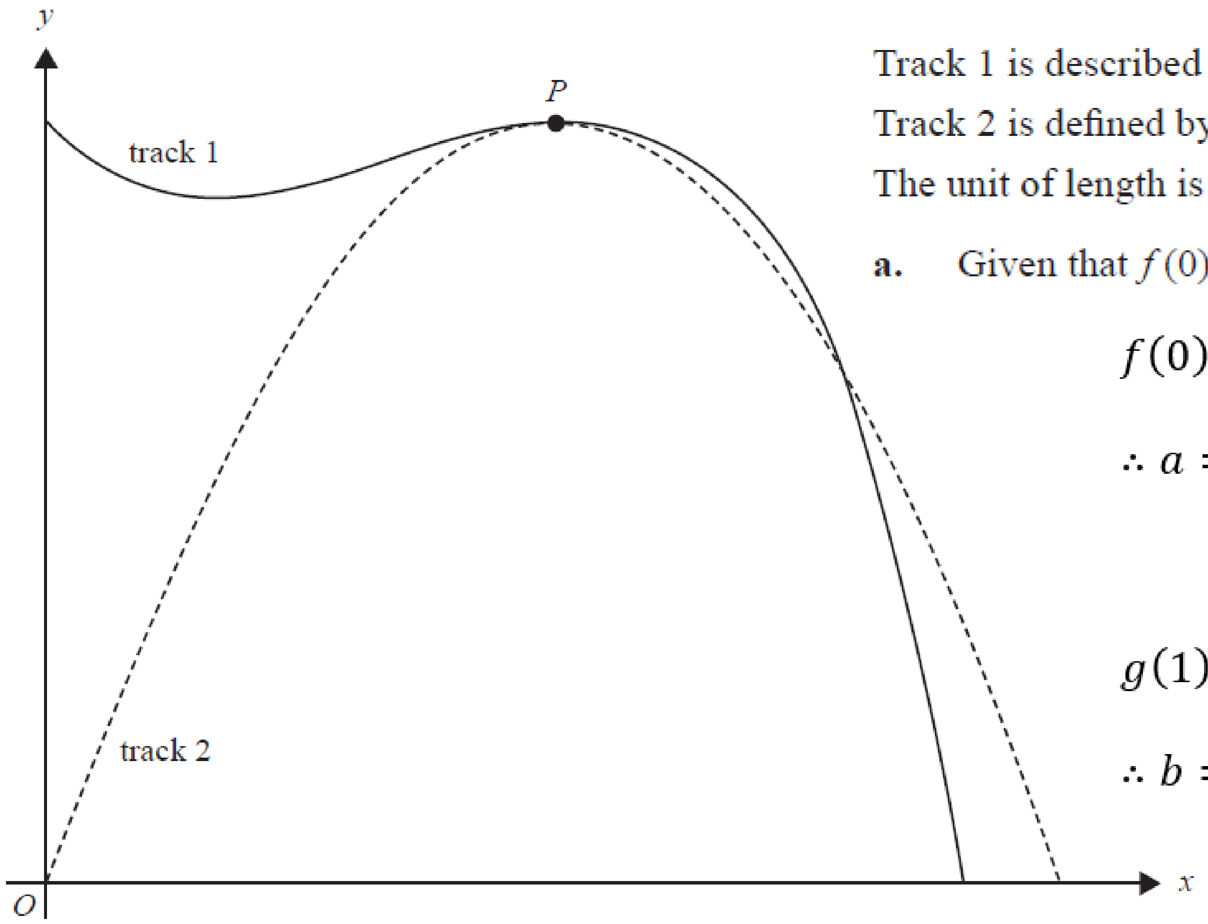
Fractions caused problems

Alternatively, can evaluate numerator and denominator separately as integrals and then have

$$\Pr(T > 2|T > 1) = \frac{\frac{9}{16}}{\frac{225}{256}} = \frac{9}{16} \times \frac{256}{225} = \frac{16}{25}$$

Question 9a.

The shapes of two walking tracks are shown below.



Marks	0	1	Average
%	10	90	0.9

Track 1 is described by the function $f(x) = a - x(x-2)^2$.

Track 2 is defined by the function $g(x) = 12x + bx^2$.

The unit of length is kilometres.

a. Given that $f(0) = 12$ and $g(1) = 9$, verify that $a = 12$ and $b = -3$.

$$f(0) = a - 0(0-2)^2 = a - 0 = 12,\ f(0) = a$$

$$\therefore a = 12$$

$$g(1) = 12 \times 1 + b \times 1^2 = 12 + b = 9$$

$$\therefore b = -3$$

Verify—so working is needed to support answers

Question 9b.

Marks	0	1	2	Average
%	34	36	30	1.0

b. Verify that $f(x)$ and $g(x)$ both have a turning point at P.
Give the co-ordinates of P.

$$f(x) = 12 - x(x-2)^2$$
$$= -x^3 + 4x^2 - 4x + 12$$
$$f'(x) = -(3x^2 - 8x + 4)$$
$$f'(2) = -(3(2)^2 - 8(2) + 4) = 0$$

$$g(x) = 12x - 3x^2$$
$$g'(x) = 12 - 6x$$
$$g'(2) = 0$$

Or

$$g(x) = 12x - 3x^2$$
$$= -3x^2 + 12x$$

Turning point at

$$x = \frac{-b}{2a} = \frac{-12}{-6} = 2$$

Maxima of the graph, either use *f* or *g*

$$f(2) = 12 - 0 = 12$$
$$(2,12)$$

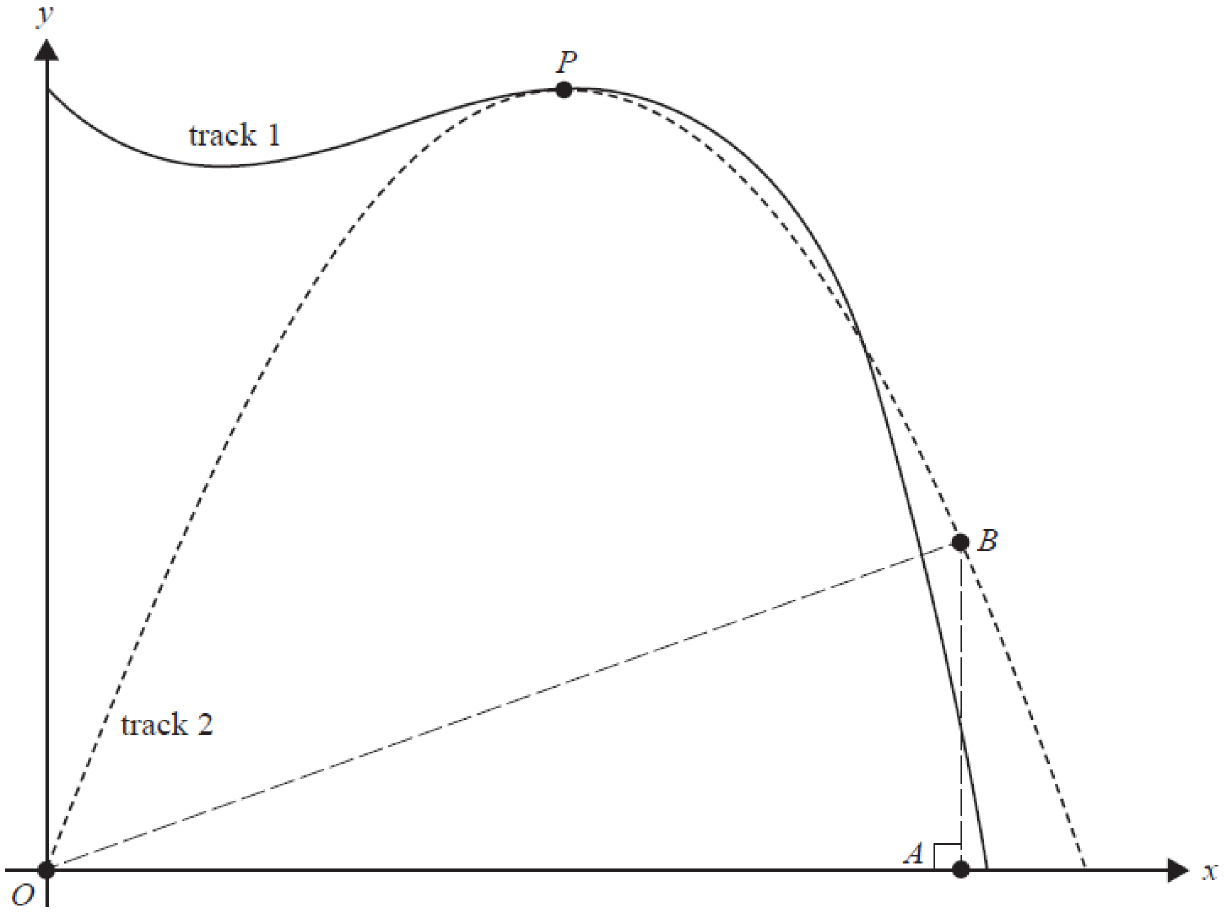
Verify question—so working needs to clearly show the intention of the question and verify both functions
Errors with finding f'(x) and with substitution

Question 9c.

Marks	0	1	2	3	Average
%	65	5	17	13	0.8

c. A theme park is planned whose boundaries will form the triangle ΔOAB where O is the origin, A is at $(k, 0)$ and B is at $(k, g(k))$, as shown below, where $k \in (0, 4)$.

Find the maximum possible area of the theme park, in km^2.



Question 9c.

Marks	0	1	2	3	Average
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c. A theme park is planned whose boundaries will form the triangle ΔOAB where O is the origin, A is at $(k, 0)$ and B is at $\left(k, g(k)\right)$, as shown below, where $k \in (0, 4)$.
Find the maximum possible area of the theme park, in km^2.

Area of triangle

$$A(x) = \frac{1}{2} \times x \times (12x - 3x^2)$$

$$= 6x^2 - \frac{3}{2}x^3$$

Maximum area at $A'(x) = 0$

$$A'(x) = 12x - \frac{9}{2}x^2 = \frac{1}{2}(24x - 9x^2) = 0$$

$$3x\left(4 - \frac{3}{2}x\right) = 0 \qquad x = \frac{24}{9} = \frac{8}{3}$$

$$A\left(\frac{8}{3}\right) = \frac{1}{2} \times \frac{8}{3} \times \left(12\left(\frac{8}{3}\right) - 3\left(\frac{8}{3}\right)^2\right)$$

$$= \frac{4}{3}\left(32 - \frac{64}{3}\right)$$

$$= \frac{1152}{81} = \frac{128}{9}$$

Set up as an area of a triangle
Use of variables
Arithmetic manipulation

The End!